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Local analytical discrete ordinate method for the solution of the radiative transfer equation

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Abstract—This paper represents a new, accurate approach to solve radiative transfer equation for each discrete ordinates. The radiative transfer equations can be approximated as a linear system of partial differential equations if the source term is averaged within the domain of a finite control volume. Then these equations can be solved analytically by superposition for multi-dimensional problems. Conventional solution (diamond differencing) of the discrete ordinate method (DOM) may lead to negative fluxes, which is physically unacceptable. The proposed method (local analytical discrete ordinate method, LADOM) does not suffer from such a problem. The predictions of LADOM are well compared with that of zonal and conventional finite-difference methods. The proposed method is more accurate, free from oscillation and simpler to apply to multi-dimensional problems than the conventional finite-difference method.

INTRODUCTION

Recently, the discrete ordinate method (DOM) is getting more attention from researchers as a simple, accurate method of solving the radiative transfer equation in combustion systems. DOM is based on converting the integro-differential radiative transfer equation into system of partial differential equations. These differential equations are compatible with general fluid flow and heat transfer codes. Such a compatibility makes the method powerful in combustion applications, where the fluid flow, heat transfer and radiation fluxes are essential in the design and control of combustion systems. The discrete ordinate method was originally developed to solve Boltzmann neutron transport equations [1], and successfully adopted to solve radiative transport equation in combustion space filled with radiatively participating media [2–4]. The efforts of the investigators can be classified into two categories, physical consideration and method of solution.

As far as the physical situation and geometry is considered, the method applied to one, two and three-dimensional rectangular and cylindrical geometries [5] filled with gray or spectrally absorbing, emitting and scattering media. Recently, the method was adopted to furnaces with obstacles, such as cooling tubes and protrusions inside the furnaces. Adams and Smith [6], simulated radiative flux distribution in a furnace with cooling tubes. Predicted fluxes were compared with experimental data and resolved highly-directional shadowing effects caused by the internal tubes. The authors concluded that the prediction accuracy depends on accurate specification of temperature profiles rather than the detailed resolution of absorption or scattering parameters. Sanchez and Smith [7], presented results for a furnace with protrusions and obstructions using the discrete ordinate method. The method is based on deactivating the domain occupied by obstructions. This technique is similar to setting high viscosity in a solution of conjugated fluid flow and heat transfer problems using the control volume approach. The technique was detailed by Chai *et al.* [8], and applied to irregular geometries.

The attention of other researchers was focused on the method of solution. The main method of solution used is based on control volume with central-difference (diamond-difference) scheme and switches to the upwind scheme when an unrealistic flux is predicted (negative flux) [3, 4, 8]. Kumar *et al.* [9] presented a method to solve the radiative transfer equation for scattering and absorbing for a one-dimensional problem. The method was based on an analytical solution of the governing equations using available two-point boundary value solver software (IMSL and NAG). But, no suggestion was made to extend the method to multi-dimensional problems. Fiveland and Jesse [10], used a finite element approximation to solve discrete ordinate equations. The method over estimates net radiative fluxes for a highly absorbing medium compared with the prediction of finite-control volume formulation and zonal method, unless fine elements are used near the boundaries. An interesting method was suggested by El Wakil and Sacadura [11], which was based on integration of the radiative transfer equation along the discrete ordinate. The integration leads to exponential variation of the intensity along that direction, instead of linear interpolation as used by the conventional finite difference method. This method is similar to the power law scheme of Patankar [12], which was suggested for the solution of convection–diffusion equations.

NOMENCLATURE

E_b emissive power, πI [W m^{-2}]
G incident energy, $\int_{4\pi} I \, d\Omega$ [W m^{-2}]
I radiant intensity [W (m$^2 \cdot$ Sr)$^{-1}$]
Q nondimensional surface heat transfer rate, G/E_b
x, y, z coordinate system [m]
X_0 defined in equation 6a
Y_0 defined in equation 6b.

Greek symbols
Φ scattering phase function
Ω, Ω' outward and inward directions of radiation intensity
δ Dirac function
ε surface emissivity
κ absorption coefficient [m^{-1}]
μ, η, ζ ordinate directions
ρ surface reflectivity
σ scattering coefficient [m^{-1}]
ω weight function in a direction.

Subscripts
b blackbody
m outgoing ordinate direction
p control volume center
e, w, s, n east, west, south and north faces of a control volume
ba, f back and front faces of a control volume.

The method presented in this paper can be simply applied to two- and three-dimensional problems. The main idea is to linearize a radiative transfer equation locally for each discrete ordinate. Then the equations can be solved by separation of variables or superposition of solutions. The method is free from oscillation and more accurate. The method is tested for two- and three-dimensional problems and results are compared with those predicted by zonal and conventional finite-difference methods.

ANALYSIS

The radiative transfer equation [13] can be written as,

$$\mu \frac{\partial I}{\partial x} + \eta \frac{\partial I}{\partial y} + \zeta \frac{\partial I}{\partial z} = -(\kappa+\sigma) I + \kappa I_b + \frac{\sigma}{4\pi} \int_{\Omega'} \Phi(\Omega' \to \Omega) I(\Omega') \, d\Omega'. \quad (1)$$

For a non gray model the intensity (I), absorption and scattering coefficients are functions of wave length.

Considering diffusive emitter and absorber boundaries, the boundary conditions are written as,

$$I_w = \varepsilon I_b + \frac{\rho}{\pi} \int_{n\Omega' < 0} | \mathbf{n} \cdot \mathbf{\Omega}' | \, I(\mathbf{\Omega}') \, d\Omega'. \quad (2)$$

The discrete-ordinates method is to transform the above integro-differential equations into a differential equation in a number of discrete angular directions, spanning the total solid angle of 4π steradiants. The discrete-ordinates of the radiative transfer equations can be written in Cartesian coordinates [5] as

$$\mu_m \frac{\partial I_m}{\partial x} + \eta_m \frac{\partial I_m}{\partial y} + \zeta_m \frac{\partial I_m}{\partial z} = -(\kappa+\sigma) I_m + \kappa I_b + S_m, \quad (3)$$

where

$$S_m = \frac{\sigma}{4\pi} \sum_j [(1-\delta_{mj}) \omega_m \Phi_{mj} I_m^j]. \quad (4)$$

Assuming that the source term (S_m) is constant within a control volume (averaged). Then equation (3) represents the set of a linear partial differential equation, which can be solved analytically. Let us assume that

$$(\kappa+\sigma) I_m - \kappa I_b - S_m = X(x) + Y(y) + Z(z). \quad (5)$$

Substitute equation (5) into equation (3) and assume that source term ($\kappa I_b + S_m$) is constant over each finite-control volume, then the solution for I_m is

$$I_m = \left(X_0 \, e^{-\frac{\kappa+\sigma}{\mu_m} x} + Y_0 \, e^{-\frac{\kappa+\sigma}{\eta_m} y} + Z_0 \, e^{-\frac{\kappa+\sigma}{\zeta_m} z} + \kappa I_b + S_m\right) / (\kappa+\sigma). \quad (6)$$

Using local coordinate for each control volume with $x = y = z = 0$ at the center of the control volume (Fig. 1), then the X_0, T_0 and Z_0 can be evaluated in terms of intensities at control volume faces.

For a two-dimensional problem Z_0 is zero and the expressions for X_0 and Y_0 are as follows:

$$X_0 = \frac{(\kappa+\sigma)\{I_{m,w} \, e^{\frac{\kappa+\sigma}{\eta} \Delta y/2} - I_{m,s}\} - (\kappa I_b + S_m)\{e^{\frac{\kappa+\sigma}{\eta} \Delta y/2} - 1\}}{e^{\frac{\kappa+\sigma}{\mu} \Delta x/2 + \frac{\kappa+\sigma}{\eta} \Delta y/2} - 1} \quad (6a)$$

and

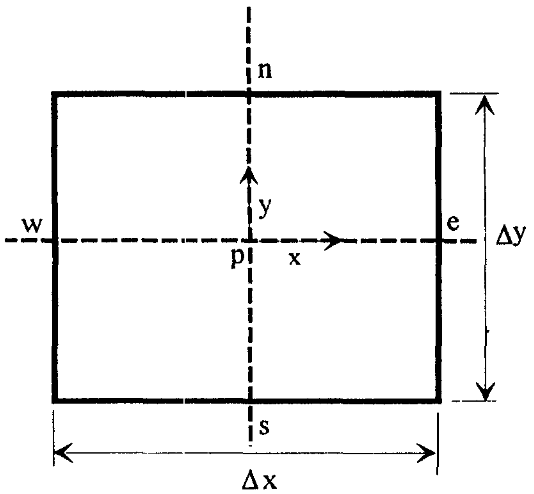

Fig. 1. Control volume.

$$Y_0 = \frac{(\kappa+\sigma)\{I_{m,s}\,e^{\frac{\kappa+\sigma}{\mu}\Delta x/2} - I_{m,w}\} - (\kappa I_b + S_m)\{e^{\frac{\kappa+\sigma}{\mu}\Delta x/2} - 1\}}{e^{\frac{\kappa-\sigma}{\eta}\Delta y/2 + \frac{\kappa+\sigma}{\mu}\Delta x/2} - 1}. \quad (6b)$$

The intensity at the center, east and north faces of the control volumes can be written as follows,

$$I_{m,p} = (X_0 + Y_0 + \kappa I_b + S_m)/(\kappa+\sigma) \quad (7a)$$

$$I_{m,e} = (X_0\,e^{-\frac{\kappa+\sigma}{\mu}\Delta x/2} + Y_0 + \kappa I_b + S_m)/(\kappa+\sigma) \quad (7b)$$

and

$$I_{m,n} = (X_0 + Y_0\,e^{-\frac{\kappa+\sigma}{\eta}\Delta y/2} + \kappa I_b + S_m)/(\kappa+\sigma), \quad (7c)$$

respectively.

The procedure of solution is to start evaluating equation (7a) for control volume at one corner of rectangular domain with given boundary conditions. Then evaluate equation 7b and 7c which will be the boundary conditions for adjacent control volumes. Repeat the procedure for other control volumes and sweep the domain of solution along x- or y-direction. The above procedure should be repeated along each ordinate. The Appendix summarizes the equations for three-dimensional geometry.

RESULTS AND DISCUSSION

The accuracy of the LADOM is examined for two- and three-dimensional rectangular enclosures. A test is carried for different values of optical thickness between 1.0 and 10.0. Emitting, absorbing and scattering mediums are tested and compared with available data. Prediction of LADOM is compared with the prediction of conventional DOM and zonal method. Previous tests [3,4] showed that S4 is a compromise between accuracy and computer time, therefore S4 is used for all predictions. Also, two sizes of control volumes compared, i.e. 10×10 and 20×20, for both LADOM and DOM method. Using a 30×30 grid size does not reveal a significant difference in the prediction of fluxes compared with predictions using 20×20 grids, unless otherwise stated. The following sections discuss results obtained by two- and three-dimensional problems, respectively.

Two-dimensional enclosure

The predictions of LADOM for two-dimensional enclosures are evaluated against predictions of conventional DOM and zonal method, for an absorbing, emitting and scattering medium. Three cases are considered:

(1) A square enclosure with unit length where all four walls are black with zero emissive powers is simulated. The medium is assigned to emissive power of unity. Figure 2a, b shows the comparison results for two values of optical thickness. For optical thickness of unity (Fig. 2a) the difference between the predictions of two methods is insignificant and well compared with the exact solution of Lockwood and Shah [14]. For optical thickness of 10.0, the DOM method

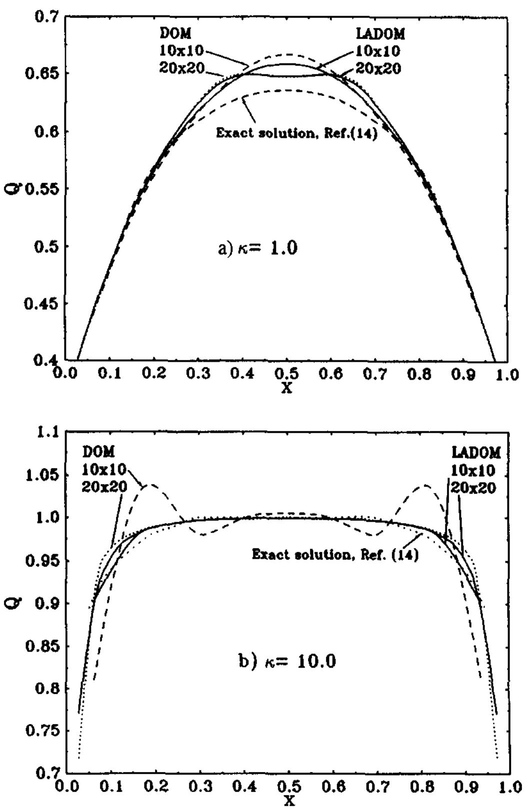

Fig. 2. Net radiative fluxes at the wall of a square cavity filled with absorbing–emitting medium, (a) optical thickness of 1.0 and (b) optical thickness of 10.0.

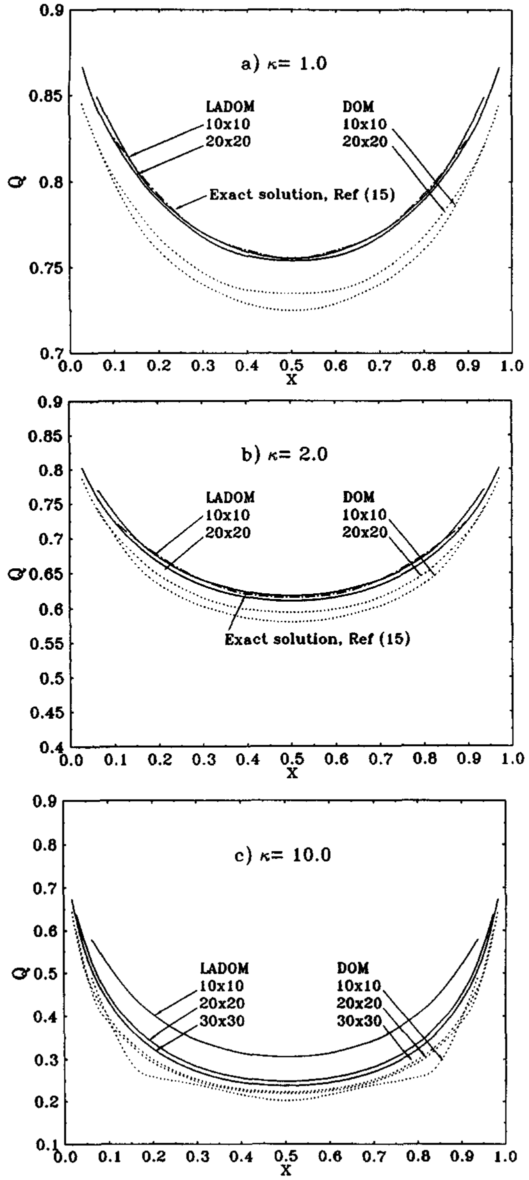

Fig. 3. Net radiative fluxes at the hot wall of a square cavity filled with absorbing–emitting medium, (a) optical thickness of 1.0, (b) optical thickness of 2.0 and (c) optical thickness of 10.0.

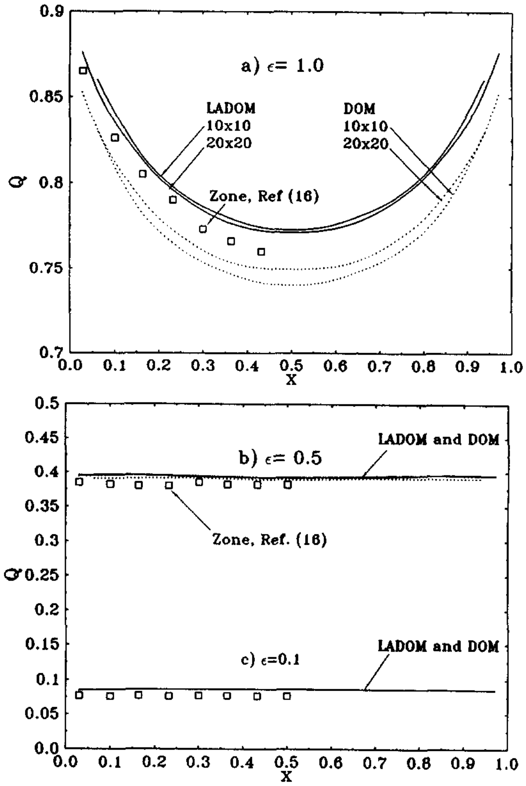

Fig. 4. Net radiative fluxes on the hot wall of square cavity filled with scattering–emitting medium (a) surface emissivity of 1.0, (b) surface emissivity of 0.5 and (c) surface emissivity of 0.1.

reveals an oscillatory prediction of the heat flux distribution for coarse control volume sizes (10×10), and oscillation diminishes by refining control volume sizes. LADOM compares well with the exact solution of ref. [14] and does not show any oscillatory predictions for coarse control volume sizes. Also, the difference between 10×10 and 20×20 control volumes is insignificant.

(2) A square enclosure with unit length where all walls are black with zero emissive powers, except one wall with emissive power of unity (hot wall) is simulated. The medium is absorbing–emitting and in radiative equilibrium. The predicted results for optical thickness of 1.0, 2.0 and 10.0 are shown in Fig. 3a, b and c, respectively. The predictions of DOM and LADOM are compared with exact solution of Razzaque *et al.* [15]. The LADOM predictions compare well with the exact solution. The difference between the prediction using 10×10 and 20×20 control volumes is not significant for optical thickness of 1.0 and 2.0, but for $\kappa = 10$ the difference is significant between the predictions of using 10×10 and 20×20 control volumes. Therefore, a 30×30 control volume is tested. For an optical thickness of 10.0 the DOM method prediction suffers from oscillatory heat flux distribution for coarse control volume, this is not the case with the prediction of LADOM.

(3) A square enclosure with unit length filled with an emitting and isotropically scattering medium (i.e. $\kappa = 0.0$, $\sigma = 1.0$) is simulated. Walls of the enclosure are gray and the different surface emittancy tested. One of the walls has an emissive power of unity (hot wall) and other walls are cold. The prediction results are obtained and compared for different emissivity of the walls surfaces (Fig. 4). This is important because the boundary condition is not only a function of the surface emissive power but of the incident radiant energy. Results of predictions using LADOM and DOM are compared with zonal results of ref. [16]. For the black body condition ($\varepsilon = 1.0$), LADOM results can be said to compare well with the zonal method. For $\varepsilon = 0.5$ and 0.1 the predictions of

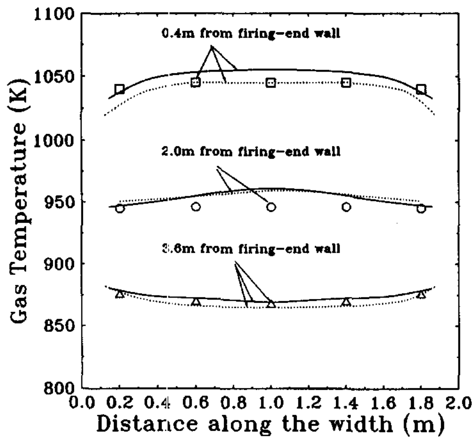

Fig. 5. Temperature distribution of the medium at three axial locations in absorbing–emitting medium.

LADOM and DOM are consistent where there is no significant difference between the predictions of both schemes.

Three-dimensional enclosure

Menguc and Viskanta [13] presented results for a three-dimensional furnace of dimensions $4.0 \times 2.0 \times 2.0$ m, with firing wall at 1200 K and with emissivity of 0.85, exit wall at 400 K with emissivity of 0.7, and other walls at 900 K with emissivities of 0.7. The optical thickness of the medium is 0.5 m^{-1}. The medium has a source power of 5.0 kW m^{-3}. Figure 5 shows the comparison of temperature distribution perdition by two methods with the results of Menguc and Viskanta. The prediction of both methods are consistent and compare well with the results of ref. [13]. Figure 6 shows the heat flux distributions at the firing and cold end walls predicted by LADOM and DOM and are compared with the results of ref. [13]. LADOM predictions are more accurate than predictions of DOM if the results of Menguc and Viskanta are assumed as a basis of comparison.

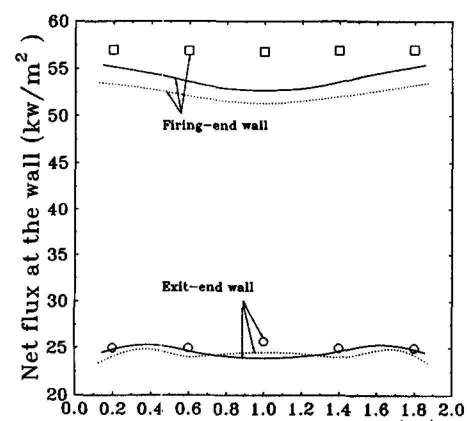

Fig. 6. Net radiative fluxes, (a) at the hot wall and (b) at the cold wall of a rectangular enclosure with source term.

CONCLUSIONS

A local analytical method is present as an alternative method of solution of the discrete-ordinate radiative transfer equations instead of the conventional finite-difference method. The method is based on an analytical solution of a multi-dimensional, linearized equation. Hence, it is more accurate than linear-interpolation of the intensity within each control volume. The tests demonstrate that LADOM does not lead to a physically unrealistic solution (negative flux). Also, LADOM does not reveal any oscillatory flux prediction in an optically thick medium. The method is in expensive computationally due to evaluation of exponential terms, but it is more accurate than the conventional finite-difference method. The test showed that 10×10 control volumes are sufficient enough to produce accurate results, except for an optical thickness of 10. Accurate results require 20×20 control volumes. The method can be applied for an emitting, absorbing and scattering medium without any difficulty.

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APPENDIX

The necessary equations for three-dimensional problem are:

$$X_0 = \frac{(\kappa I_b + S_m)(g+h-gh-1) + (\kappa+\sigma)[I_{m,f}(1-g) + I_{m,s}(1-h) + I_{m,w}(gh-1)]}{2-f-g-h-fgh}$$

$$Y_0 = \frac{(\kappa I_b + S_m)(f+h-fh-1) + (\kappa+\sigma)[I_{m,f}(1-f) + I_{m,s}(hf-1) + I_{m,w}(1-h)]}{2-f-g-h-fgh}$$

$$Z_0 = \frac{(\kappa I_b + S_m)(f+g-fg-1) + (\kappa+\sigma)[I_{m,f}(fg-1) + I_{m,s}(1-f) + I_{m,w}(1-g)]}{2-f-g-h-fgh},$$

where

$$f = e^{\frac{\kappa+\sigma}{\mu}\Delta x/2}, \quad g = e^{\frac{\kappa+\sigma}{\mu}\Delta y/2} \quad \text{and} \quad h = e^{\frac{\kappa+\sigma}{\mu}\Delta z/2}.$$

The radiative intensities at the center, east, north and back faces of the control volume are

$$I_{m,p} = (X_0 + Y_0 + Z_0 + \kappa I_b + S_m)/(\kappa+\sigma),$$

$$I_{m,e} = (X_0\, e^{\frac{-\kappa+\sigma}{\mu}\Delta x/2} + Y_0 + Z_0 + \kappa I_b + S_m)/(\kappa+\sigma),$$

$$I_{m,n} = (X_0 + Y_0\, e^{\frac{-(\kappa+\sigma)}{\eta}\Delta y/2} + Z_0 + \kappa I_b + S_m)/(\kappa+\sigma),$$

and

$$I_{m,ba} = (X_0 + Y_0 + Z_0\, e^{\frac{-(\kappa+\sigma)}{\zeta}\Delta z/2} + \kappa I_b + S_m)/(\kappa+\sigma), \quad \text{respectively.}$$